Model Discrimination via Designed Experiments: Discriminating between the Terminal and Penultimate Models on the Basis of Composition Data

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ABSTRACT: In this paper, computer simulations have been used to study the application of statistical model discrimination methods to the modeling of copolymerizations. Statistical model discrimination methods describe how experiments should be designed and analyzed to obtain the maximum possible information on the strengths of competing models. The potential benefits of applying model discrimination methods are described by contrasting them with past work on the modeling of copolymerizations. Three model discrimination techniques (Buzzi-Ferraris and Forzatti,[15] exact entropy,[14] and Hsiang and Reilly[21]) have been applied to the systems styrene–acrylonitrile, styrene–methyl methacrylate, and styrene–butyl acrylate. Simulated copolymer composition data were used to discriminate between the terminal and penultimate models for these systems. Data are presented to show that the computer simulations are capable of accurately predicting copolymer composition, sequence distribution, and rate data. The results of the simulations show that model discrimination techniques, particularly the Buzzi-Ferraris and Forzatti[15] method, are capable of discriminating between the terminal and penultimate models on the basis of copolymer composition data. The Buzzi-Ferraris method is capable of deciding between the two models in fewer experiments than have been performed in past research and is capable of detecting explicit penultimate effects which are smaller than that found by Hill et al.[30] for the system styrene–acrylonitrile. This work on copolymer composition suggests that model discrimination methods will improve the modeling of copolymer systems and that composition data may be more useful in discriminating between copolymer models than previously thought.

Introduction

Several models have been proposed to describe free-radical copolymerization.[1,2] Researchers have fit these models to various copolymer characteristics such as composition, average sequence length, and rate, in an attempt to find the "best" copolymerization model for specific systems. When traditional rate measurements seemed to be inadequate, new experimental techniques were developed to provide additional information.[3-5] However, at this point there is still an ongoing discussion about the validity of proposed models and the effectiveness of available measurements.[6-11]

Each of the copolymerization models stems from a mechanism and it is important to distinguish between a "mechanistic model" and the underlying "mechanism". A mechanistic model is the mathematical representation of the mechanism that is used to predict characteristics such as reaction rates and product compositions. There will almost always be a difference between the mechanism and the mechanistic model because assumptions are usually invoked to make the mathematical representation more tractable.

There are two general approaches to finding the "best" mechanistic model for a free radical copolymerization. The first approach is to perform an experiment which is designed to verify a particular step or steps in the reaction mechanism. An example of this approach is the work done by Jones et al.[12] to show that a penultimate effect could exist in free radical copolymerization. The second approach is to analyze the fit of proposed models to data. However, the analysis is only as valuable as the experimental data set that has been collected. Statistical model discrimination methods are used to choose experimental conditions so that each experiment will contain the maximum possible amount of information on the strengths and weaknesses of competing models.[13,14] Statistical model discrimination methods also describe how to reliably analyze the data to determine which model(s) provide the "best" description of the data. Finding the model that provides the best description of a copolymerization is important in the design and control of copolymerization processes. In addition, if the modeling assumptions are correct, finding the model which provides the best description of experimental data will help identify the most plausible mechanism.

The application of statistical model discrimination techniques should improve our ability to discriminate between competing copolymer models. However, to the best of the authors' knowledge, statistical model discrimination techniques have not been applied to the problem of discriminating between free-radical copolymerization models. The objectives of our work are to investigate the benefits of applying these methods to a wide range of copolymerization systems, to compare the effectiveness of available measurements (composition, triad fractions, polymerization rate), and to compare some of the methods that have been proposed in the statistical literature. These objectives are very broad and general. They require the study of more than one copolymer system and the testing of more than one method for designing experiments. It would take years to meet these objectives through a purely experimental study.

In order to address these broad objectives in a reasonable amount of time, computer simulations were proposed for the first stage of the research. Computer programs were developed which can design model discrimination experiments, simulate laboratory experiments, and analyze the resulting data. The key to this process is the simulation of experimental data. A great deal of effort has been invested to ensure that the simulated data are representative of the data that would be obtained in the laboratory.

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Therefore, the simulation results should provide the same "type of information" that would be obtained by testing model discrimination methods using laboratory experiments.

This paper describes the first set of simulation studies in which statistical model discrimination methods were used to discriminate between the terminal and penultimate models for copolymerization based on simulated copolymer composition data. The paper begins with an explanation of why model discrimination methods should improve the modeling of copolymerizations. Statistical model discrimination theory is briefly reviewed, followed by a description of the three methods that were tested in the simulations. The terminal and penultimate models are then developed to clarify the equations that were used to fit simulated data. Past work on the problem of discriminating between the terminal and penultimate models is then contrasted with statistical model discrimination methods to show the potential benefits of applying these techniques. In the remainder of the paper we describe the first set of simulation studies from our research based on composition measurements.

Review of Statistical Model Discrimination Methods

General Theory. The problem of model discrimination occurs when m rival models have been proposed to describe a system and it is not certain which of the m models is "best". The models are of the form

$$y_i = f_i(\mathbf{x},\theta_i) + \epsilon_i \qquad i = 1, 2, ..., m \tag{1}$$

where y_i is the measured variable, $\mathbf{x}$ the vector of experimental conditions, θ_i the vector of model parameters, and ϵ_i the experimental error. For example, for copolymerization, the experimental conditions include feed composition, and the parameters are the reactivity ratios. The form of the model, f_i, is generally nonlinear (in the variables and the parameters) and the distribution of the error ϵ_i may not be known.

The first step in model discrimination is to obtain the $k_i x1$ vector of parameter estimates $\hat{\theta}_i$ for each of the m models. Parameter estimates can often be obtained from the literature, or from a researcher's past work, but these parameter estimates may not be as precise as they could be, unless the experiments were designed specifically for the purpose of parameter estimation. For this reason most model discrimination methods involve a series of initial experiments designed for the purpose of parameter estimation. For the ith model, the number of parameter estimation experiments must be equal or greater than k_i. For the parameters in all m models to be estimated, the number of parameter estimation experiments must be equal to or greater than the largest value of k_i. Once the parameters have been estimated, model predictions, $\hat{y}_i$, can be calculated and used in the design of model discrimination experiments. The variance of individual predictions, var($\hat{y}_i$), can also be calculated on the basis of the assumed distribution of the errors.

The process of discriminating between competing models is typically a sequential process which involves repeating two steps. The first step is to maximize a discrimination criterion which is used to design the next set of experimental conditions. After the experiment is performed, the data are analyzed and a stopping rule is used to decide if one model can be judged significantly better than the rest. If none of the models can be deemed "best" then the process is repeated. There are different approaches to both the design and analysis steps.

Most of the discrimination criteria which have been proposed have a common basis. For any experiment it is expected that the "best" model will provide the most accurate prediction of the observed value. Therefore, it seems logical that the "best" model would be most easily identified by performing the next experiment at the point $\mathbf{x}_n$ where the average difference in model predictions is largest. The experimental point $\mathbf{x}_n$ should be chosen using the criterion

$$\max \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} (\hat{y}_i - \hat{y}_j)^2 \tag{2}$$

Normally, only a finite number of experimental conditions is considered in order to make the problem of searching for the maximum more tractable. If the points considered are evenly spaced over the range of possible $\mathbf{x}_n$ values and the number of points is large, the results will approximate a continuous search on the $\mathbf{X}$ space.

The problem of analyzing the resulting data lends itself to a Bayesian analysis. After obtaining initial parameter estimates, prior probabilities are assigned which express our belief in the m models. The prior probability of the ith model is represented by $\Pr(M_i)$, and the total probability is restricted such that

$$\sum_{i=1}^{m} \Pr(M_i) = 1 \tag{3}$$

If we have no previous knowledge about the strengths of the models, then all prior probabilities should be equal.

$$\Pr(M_i) = \frac{1}{m} \qquad i = 1, 2, ..., m \tag{4}$$

However, if previous research or theory suggests model j is more reasonable than the rest, $\Pr(M_j)$ should be set higher than the other $\Pr(M_i)$. The ratio $\Pr(M_j)/\Pr(M_i)$ should reflect the strength of our belief that model j is a better model for the system than model i. After a model discrimination experiment has been carried out, the prior probabilities are updated according to Bayes theorem:

$$\Pr(M_{i,n}) = \Pr(M_{i,n-1}) L(M_i|y_n) \tag{5}$$

where $L(M_i|y_n)$ is the likelihood of model i given the observation y_n. The prior probabilities once updated are referred to as posterior probabilities, but it is simpler to use the subscripts $n-1$ and n to identify the model probabilities before and after the nth experiment. This process of picking a new experimental point and updating the model probabilities is continued until the prior probability of one model, for example model j, $\Pr(M_j)$, becomes sufficiently large that model j is judged to be the "best". The best model is subsequently examined to ensure that it is an adequate description of the system.

Several non-Bayesian analysis methods have also been used. F-statistics and likelihood ratios have been used to compare the fit of competing models.[15-17] For more than two models, all models would normally be compared to the model which provides the best fit in order to determine if this fit is significantly better than that provided by the other models. For nested models, the values of both the F-statistic and the likelihood ratio have known distributions. This allows the development of a stopping rule based on the confidence level associated with the F-statistic or likelihood ratios. If the models are nonnested, the problem of developing a stopping rule becomes more complex but work has been done in this area.[18] In addition, Broucek et al.[19] have suggested the use of the ratio of standard deviations $\hat{\sigma}_{\min}/\sigma_i$ to compare the adequacy of

competing models, and Akaike[20] has proposed an information criterion for assessing model adequacy.

The following subsections contain descriptions of the three basic model discrimination methods investigated in our simulation studies. These are the exact entropy criterion proposed by Reilly,[14] the method of Hsiang and Reilly,[21] and the Buzzi-Ferraris et al.[15,22,23] criterion.

Exact Entropy Method. Reilly[14] developed the exact entropy criterion based on the earlier method of Box and Hill.[24] Box and Hill[24] introduced the use of entropy S, where

$$S = -\sum_{i=1}^{m} \Pr(M_i) \ln \Pr(M_i) \tag{6}$$

The maximum entropy value occurs when there is no information as to which model is best, $\Pr(M_i) = 1/m$, and the minimum value occurs when it is certain that model j is best, $\Pr(M_j) = 1$. Therefore the nth experiment should be chosen to maximize the change in S. By using inequalities to simplify the expression for the change in entropy, Box and Hill[24] derived the maximum possible change in entropy D, which is given by

$$D = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=i+1}^{m} \Pr(M_{i,n-1})\Pr(M_{j,n-1}) \left\{ \frac{(\sigma_i^2 - \sigma_j^2)^2}{(\sigma^2 + \sigma_i^2)(\sigma^2 + \sigma_j^2)} + (\hat{y}_{n,i} - \hat{y}_{n,j})^2 \left(\frac{1}{\sigma^2 + \sigma_i^2} + \frac{1}{\sigma^2 + \sigma_j^2} \right) \right\} \tag{7}$$

where σ^2 is the variance and σ_i^2 has been used to represent the prediction variance, $\mathrm{var}(\hat{y}_i)$. The variance σ^2 is never known with certainty. It is generally replaced by an estimate obtained from replicate experiments or the smallest estimate s^2 produced by regressing the m models.

Reilly[14] showed that numerical methods could be used to calculate the exact value of the change in entropy R, which is given by

$$R = -\frac{1}{2}(1 + \ln 2\pi) - \sum_{i=1}^{m} \Pr(M_{i,n-1}) \left\{ \frac{1}{2} \ln(\sigma^2 + \sigma_i^2) + E_i(\ln q(y_n)) \right\} \tag{8}$$

where $E_i(\ln q(y_n))$ is the expected value of $\ln q(y_n)$ assuming model i to be correct. The function $q(y_n)$ is defined as

$$q(y_n) = \sum_{i=1}^{m} \Pr(M_{i,n-1}) p_i \tag{9}$$

where p_i is the normal probability of observation y_n based on the ith model. The value of $E_i(\ln q(y_n))$ in eq 8 can be evaluated using a Gauss–Hermite quadrature. A Gauss–Hermite quadrature can be used to evaluate the expected value of a function $f(w)$ where w is a variable with mean μ and variance ν.

$$E|f(w)| = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{n} H_k f(\mu + z_k \sqrt{2\nu}) \tag{10}$$

H_k and z_k are the weights and zeroes of the nth order Gauss–Hermite quadrature. Assuming model i to be correct, y_n has an estimated mean of $\hat{y}_i$ and a variance of $\sigma^2 + \sigma_i^2$. Therefore eq 10 becomes

$$E_i(\ln q(y_n)) = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{n} H_k \ln\left(\sum_{j=1}^{m} \Pr(M_j) p_j \right) \tag{11}$$

where

$$p_j = \frac{1}{\sqrt{2\pi(\sigma^2 + \sigma_j^2)}} \times \exp\left(\frac{-1}{2(\sigma^2 + \sigma_j^2)} (\hat{y}_i + z_k \sqrt{2(\sigma^2 + \sigma_i^2)} - \hat{y}_j)^2 \right) \tag{12}$$

Substitution of eq 11 into eq 8 allows the evaluation of R. Model discrimination experiments can then be designed by finding the point $\mathbf{x}_n$ which maximizes the value of R.

After the nth experiment is performed, the prior probabilities are updated using

$$\Pr(M_{i,n}) = \frac{\Pr(M_{i,n-1}) L(M_i|y_n)}{\sum \Pr(M_{i,n-1}) L(M_i|y_n)} \tag{13}$$

where $L(M_i|y_n)$ is the likelihood of the ith model based on the observation y_n

$$L(M_i|y_n) = \frac{1}{\sqrt{2\pi(\sigma^2 + \sigma_i^2)}} \exp\left(-\frac{1}{2(\sigma^2 + \sigma_i^2)} (y_n - \hat{y}_{i,n})^2 \right) \tag{14}$$

The exact entropy method considers both the average difference in model predictions and prediction variance. Calculation of prediction variance σ_i^2 requires linearization of the model about $\theta_i = \hat{\theta}_i$. The prediction variance is calculated using the equation

$$\sigma_i^2 = \mathbf{F}(\mathbf{x}_n)(\mathbf{F}'\mathbf{F})_{n-1}^{-1}\mathbf{F}(\mathbf{x}_n)'\sigma^2 \tag{15}$$

where $\mathbf{F}$ is the Jacobian matrix based on the first $n-1$ experiments and $\mathbf{F}(\mathbf{x}_n)$ is the Jacobian evaluated at the new point $\mathbf{x}_n$. The general element of the Jacobian matrix for the ith model can be written as

$$F_{jk} = \left. \frac{\partial f_i(\mathbf{x},\theta_i)}{\partial \theta_k} \right|_{\mathbf{x}=\mathbf{x}_j,\theta_i=\hat{\theta}_i} \tag{16}$$

where θ_k is the kth element of the vector θ_i.

Hsiang and Reilly Criterion. Hsiang and Reilly[21] developed a method which is different from the exact entropy method. In the exact entropy method uncertainty in the parameter estimates and the model predictions is estimated by linearizing the models at the values of the point estimates. In the Hsiang and Reilly[21] method, uncertainty in the parameter estimates is expressed in the form of discrete parameter probability distributions. The symbol $\tilde{\theta}_k$ is used to represent a particular combination of parameter values being considered for the ith model.

The first step in the method is to assign prior probabilities, both to the models and to all parameter vectors denoted by $\tilde{\theta}_i$. Prior model probabilities $\Pr(M_i)$ are assigned on the basis of our belief that the ith model is correct. Similarly, prior probabilities for parameter values are assigned on the basis of previous knowledge. For model i, a range of values with nonnegligible probability is defined for each parameter. Each parameter is then represented by a reasonable number of discrete values over the range, for example 10–50 values. The probability corresponding to a specific combination of the discrete values is represented by $\Pr(\tilde{\theta}_i|M_i,y)$ and the set of prior parameter probabilities is stored in a matrix. For example, consider the two parameter model:

$$y = \theta_1 + \theta_2 x_1 + \epsilon \tag{17}$$

The parameter θ_1 can be represented by a discrete values and the parameter θ_2 can be represented by b discrete values. The probabilities for all combinations of θ_1 and θ_2 are stored in an $a \times b$ matrix. For a given row of the

matrix, θ_1 is constant and the values represent the conditional probability distribution of θ_2. For a given column of the matrix, θ_2 is constant and the values represent the conditional probability distribution of θ_1. After values have been assigned to reflect the conditional distributions of the parameters, they are adjusted so that the total probability is one. The m rival models are written in the form

$$y_i = f_i(\mathbf{x},\theta_i) + \epsilon_i \qquad i = 1, 2, ..., m \tag{18}$$

The errors are assumed to have mean zero and the form of the distribution is assumed known (e.g. normal, γ, uniform etc.), although the error variance, σ^2, may not be known. From the error distribution, the distribution of y is derived, which in turn allows the calculation of the model likelihoods $L(M_i|\tilde{\theta}_i,y)$.

The discrimination criterion states that the nth experiment should be chosen to maximize

$$C = \sum_{i=1}^{m}\sum_{j=i+1}^{m} |\hat{y}_i - \hat{y}_j|[\Pr(M_{i,n-1}) + \Pr(M_{j,n-1})] \tag{19}$$

where $\hat{y}_i$ is the expected value of y given by model i. The value of $\hat{y}_i$ is calculated by taking the expectation of y_i over all possible values of the parameters and is written as

$$\hat{y}_i = \sum_{\theta_i} f_i(x_n,\tilde{\theta}_i)\Pr(\tilde{\theta}_i|M_i,y) \tag{20}$$

Once the nth experiment has been carried out, the model probabilities are updated using

$$\Pr(M_{i,n}) = \frac{\Pr(M_{i,n-1})\sum_{\theta_i}\Pr(\tilde{\theta}_i|M_i)\mathrm{L}(M_i|\tilde{\theta}_i,y_n)}{\sum_{i=1}^{m}\Pr(M_{i,n-1})\sum_{\theta_i}\Pr(\tilde{\theta}_i|M_i)\mathrm{L}(M_i|\tilde{\theta}_i,y_n)} \tag{21}$$

and the parameter probabilities are updated using

$$\Pr(\tilde{\theta}_i|M_i,y_n) = \frac{\Pr(\tilde{\theta}_i|M_i)\mathrm{L}(M_i|\tilde{\theta}_i,y_n)}{\sum_{\theta_i}\Pr(\tilde{\theta}_i|M_i)\mathrm{L}(M_i|\tilde{\theta}_i,y_n)} \tag{22}$$

where the likelihood $\mathrm{L}(M_i|\tilde{\theta}_i,y_n)$ is given by

$$\mathrm{L}(M_i|\tilde{\theta}_i,y_n) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(y_n - f_i(x_n,\tilde{\theta}_i))^2\right) \tag{23}$$

After the probabilities have been updated, some parameter combinations may have negligible probabilities. The matrix of parameter probabilities may be rescaled to eliminate combinations with negligible probability, thus reducing computer storage requirements.

The Hsiang and Reilly[21] method has not been used widely in engineering studies because of its need for a large amount of computer storage space. However, with the advances in computers that have occurred since the method was introduced, this should no longer be a constraint.

The method has two major advantages. The first is that all previous knowledge of the parameters can be included in the form of probability distributions. In addition, parameter information can be updated and prediction variances can be calculated without linearizing the model. This is an improvement over other methods which assume a normal distribution characterized by the point estimate $\hat{\theta}_i$. In addition, σ can be treated as a parameter. This is more realistic than using a point estimate because σ is never known in real situations.

Buzzi-Ferraris Criterion. Although the Hsiang and Reilly[21] criterion is different from the exact entropy[14] criterion, they are both based on a Bayesian analysis of the data. An alternative approach was taken by Buzzi-Ferraris et al.[15,22,23]

Buzzi-Ferraris and co-workers proposed that nth experiment be chosen to maximize

$$T = \frac{\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(\hat{y}_i - \hat{y}_j)^2}{(m-1)(ms^2 + \sum_{i=1}^{m} s_i^2)} \tag{24}$$

An independent estimate of variance s^2 is used instead of σ^2, and the estimated prediction variance s_i^2 is used instead of σ_i^2. (Since the terminal model is nested within the penultimate model, we have used the pentultimate regression estimate of variance s_{pent}^2 as the best estimate of σ^2.) This criterion is a simple ratio of the variance in the difference between predictions to the average variance in the predictions. Since the form of criterion T is similar to an F-statistic, the value of T is used to judge the ability to discriminate between the models. If it is not possible to discriminate between models at a given experimental condition, the value of T will be less than one. If the experimental error is large relative to the differences between predictions, it may not be possible to discriminate between rival models, and T will be less than one at all possible experimental points. This type of information is useful for recognizing when several models are equally good descriptions of a system or, alternatively, that improved parameter estimates are needed to achieve discrimination. Therefore, the maximum value of T can be used to decide whether the next experiment should be a model discrimination experiment or whether parameter estimation experiments are needed. This property of the Buzzi-Ferraris criterion makes it simpler to implement than the other criteria discussed, which do not provide information on whether or not discrimination can be achieved. They are based on the implicit assumption that one model provides a better description of the system, and any decision to switch to parameter estimation experiments is based solely on the experimenter's judgement.

The Buzzi-Ferraris method also takes a different approach to data analysis. In their first paper, Buzzi-Ferraris and Forzatti[15] suggested that the adequacy of all competing models should be checked and only models shown to be adequate should be used in the next evaluation of T. To check the adequacy of each model, they proposed an F-test on the variance estimate s^2 produced by regression and some independent estimate $\hat{\sigma}^2$. Alternatively, they suggested a Hartley F max test[25] or Bartlett's χ^2 test.[26] However, since the terminal and penultimate models are nested, we have chosen to use a general form of the F-test which is used with nested models. For example, for a pair of nested models, where model 2 is nested in model 1, the F statistic is

$$F = \frac{(\mathrm{SSR}_2 - \mathrm{SSR}_1)/(k_1 - k_2)}{\mathrm{SSR}_1/n - k_1} \tag{25}$$

where SSR is the sum of squared residuals produced by a regression analysis. The F statistic is compared to the F distribution with $(k_1 - k_2)$ and $(n - k_1)$ degrees of freedom to determine whether the extra parameters in model 1

provide a significantly better fit to the data. Only models judged to be an adequate description of the system are used in the next evaluation of T. Buzzi-Ferraris et al.[22,23] suggest this will eliminate the negative effects of poor variance estimates on the Bayesian analysis. However, some caution is necessary since F tests are absolute measures of model adequacy only if a good estimate of the experimental error is available.

Copolymerization Models

In the current work, the focus is on discrimination between the terminal and penultimate models. Both the terminal and penultimate models were developed to describe the reaction of two monomers M_1 and M_2 to form a copolymer. For a free-radical polymerization the growing radical of length n is represented by $R_n^{\bullet}$.

The terminal model assumes only the last unit on the growing chain influences monomer addition. If $R_{n,i}^{\bullet}$ is a radical ending in monomer i, the four propagation reactions can be summarized as

$$R_{n,i}^{\bullet} + M_j \xrightarrow{k_{ij}} R_{n+1,j}^{\bullet} \tag{26}$$

where the first subscript on R denotes the radical chain length, i and j are used to denote monomer units (1 or 2), and k_{ij} is the rate constant for addition of monomer j to a radical ending in unit i. Although there are four propagation rate constants k_{ij}, the polymerization rate is usually expressed in terms of the two homopolymerization rate constants, k_{11} and k_{22}, and two monomer reactivity ratios

$$r_1 = k_{11}/k_{12} \qquad r_2 = k_{22}/k_{21} \tag{27}$$

The penultimate model assumes that the last two monomer units on the growing radical chain influence monomer addition. If $R_{n,ij}^{\bullet}$ is a radical with terminal unit j and penultimate unit i, the eight propagation reactions can be summarized as

$$R_{n,ij}^{\bullet} + M_k \xrightarrow{k_{ijk}} R_{n+1,jk}^{\bullet} \tag{28}$$

The rate of polymerization is usually expressed in terms of the two homopolymerization rate constants k_{111} and k_{222}, along with four monomer reactivity ratios

$$r_{11} = k_{111}/k_{112} \qquad r_{21} = k_{211}/k_{212}$$
$$r_{22} = k_{222}/k_{221} \qquad r_{12} = k_{122}/k_{121} \tag{29}$$

and two radical reactivity ratios

$$s_1 = k_{211}/k_{111} \qquad s_2 = k_{12\,2}/k_{222} \tag{30}$$

The terminal model is nested within the penultimate model. One can see that if an attempt is made to fit the penultimate model to a system best described by the terminal model, joint confidence regions for the reactivity ratio estimates will include the points

$$\hat{r}_{11} = \hat{r}_{21}$$
$$\hat{r}_{22} = \hat{r}_{12} \tag{31}$$
$$\hat{s}_1 = \hat{s}_2 = 1$$

Equations for instantaneous composition, triad fractions, and propagation rate constants can be derived on the basis of the reactions given in eqs 26 and 28.

Copolymer Composition. The instantaneous composition, in terms of the mole fraction of monomer 1 bound in the copolymer, is denoted by F_1. The terminal model prediction is given by

$$F_1 = \frac{r_1 f_1^{\,2} + f_1 f_2}{r_1 f_1^{\,2} + 2 f_1 f_2 + r_2 f_2^{\,2}} \tag{32}$$

where f_1 and f_2 are the mole fractions of unbound (free) monomer 1 and 2. The penultimate model prediction is given by

$$F_1 = \frac{r_{21} f_1^{\,2}\left(\dfrac{r_{11} f_1 + f_2}{r_{21} f_1 + f_2}\right) + f_1 f_2}{r_{21} f_1^{\,2}\left(\dfrac{r_{11} f_1 + f_2}{r_{21} f_1 + f_2}\right) + 2 f_1 f_2 + r_{12} f_2^{\,2}\left(\dfrac{r_{22} f_2 + f_1}{r_{12} f_2 + f_1}\right)} \tag{33}$$

However, at conversions above 1 or 2 mol % it would be preferable to use the cumulative composition. The integrated composition equation tracks the change in feed composition with conversion which allows us to calculate cumulative composition. The change in feed composition with mole conversion (x) is defined by

$$\ln(1 - x) = \int_{f_{1,0}}^{f_1} \frac{df_1}{F_1 - f_1} \tag{34}$$

For the terminal model this integrates to the Meyer–Lowry equation which is written as

$$x = 1 - \left(\frac{f_1}{f_{1,0}}\right)^{\alpha}\left(\frac{1 - f_1}{1 - f_{1,0}}\right)^{\beta}\left(\frac{f_{1,0} - \delta}{f_1 - \delta}\right)^{\gamma} \tag{35}$$

where

$$\alpha = \frac{r_2}{1 - r_2} \qquad \beta = \frac{r_1}{1 - r_1}$$
$$\gamma = \frac{1 - r_1 r_2}{(1 - r_1)(1 - r_2)} \qquad \delta = \frac{1 - r_2}{2 - r_1 - r_2} \tag{36}$$

However, when we substitute the penultimate expression for F_1 into eq 34, there is no analytical solution and the equation must be solved numerically.

For both the terminal and penultimate models the integrated composition equation can be solved for f_1 given estimates of the reactivity ratios, the initial feed composition $f_{1,0}$, and mole conversion x. The cumulative composition of the copolymer can then be expressed as

$$\bar{F}_1 = \frac{f_{1,0}}{x} - \frac{(1 - x) f_1}{x} \tag{37}$$

Triad Fractions. Monomer 1 centered triads are denoted by A_{111}, A_{211}, A_{112}, and A_{212}. The terminal model predictions are given by[27]

$$A_{111} = \frac{r_1^{\,2} f_1^{\,2}}{r_1^{\,2} f_1^{\,2} + 2 r_1 f_1 f_2 + f_2^{\,2}} \tag{38}$$

$$A_{211} = A_{112} = \frac{r_1 f_1 f_2}{r_1^{\,2} f_1^{\,2} + 2 r_1 f_1 f_2 + f_2^{\,2}} \tag{39}$$

$$A_{212} = \frac{f_2^{\,2}}{r_1^{\,2} f_1^{\,2} + 2 r_1 f_1 f_2 + f_2^{\,2}} \tag{40}$$

The penultimate predictions are given by

$$A_{111} = \frac{r_{21}r_{11}f_1^2}{r_{21}r_{11}f_1^2 + 2r_{21}f_1f_2 + f_2^2} \quad (41)$$

$$A_{211} = A_{112} = \frac{r_{21}f_1f_2}{r_{21}r_{11}f_1^2 + 2r_{21}f_1f_2 + f_2^2} \quad (42)$$

$$A_{212} = \frac{f_2^2}{r_{21}r_{11}f_1^2 + 2r_{21}f_1f_2 + f_2^2} \quad (43)$$

The expressions for monomer 2 centered triads can be obtained by switching subscripts 1 and 2 in eqs 38–43.

Copolymerization Rate. The terminal model prediction of the overall propagation rate constant k_p, is given by

$$k_p = \frac{r_1f_1^2 + 2f_1f_2 + r_2f_2^2}{\frac{r_1f_1}{k_{111}} + \frac{r_2f_2}{k_{222}}} \quad (44)$$

and the penultimate prediction is written as

$$k_p = \frac{r_{21}f_1^2\left(\frac{r_{11}f_1 + f_2}{r_{21}f_1 + f_2}\right) + 2f_1f_2 + r_{12}f_2^2\left(\frac{r_{22}f_2 + f_1}{r_{12}f_2 + f_1}\right)}{\frac{r_{12}f_2}{k_{222}}\left(\frac{r_{22}f_2 + f_1/s_2}{r_{12}f_2 + f_1}\right) + \frac{r_{21}f_1}{k_{111}}\left(\frac{r_{11}f_1 + f_2/s_1}{r_{21}f_1 + f_2}\right)} \quad (45)$$

Several experimental techniques have been developed to estimate the overall propagation rate constant (k_p) from rate data. The rotating sector method is described by Stickler,[28] the spatially intermittent polymerization (SIP) method was developed by O'Driscoll and Mahabadi,[3] the electron spin resonance (ESR) method has been used by Ballard et al.[5] as well as by Lau et al.,[29] and the pulsed laser method was proposed by Olaj et al.[4] The pulsed laser and ESR methods seem to hold the most promise. Both methods involve a simple model that relates the experimental measurement to the overall propagation constant.

In the pulsed laser method, polymerization is initiated inside ampules by intermittent pulses of laser light. The length of the pulse is t_L and the length of the dark time between pulses is t_D, with the dark time being much greater than the length of the pulse ($t_D \gg t_L$). The chain length (υ_p) formed between pulses is given by

$$\nu_p = k_p t_D[M] \quad (46)$$

where [M] is the average monomer concentration.

Previous Research in the Area of Copolymer Model Discrimination

To the best of the authors' knowledge, statistical model discrimination methods have not been applied to the problem of discriminating between the terminal and penultimate models. A review of the copolymerization literature has been performed in order to show that the application of statistical model discrimination techniques has the potential to improve discrimination between the terminal and penultimate models. Most existing research shares common characteristics with respect to the planning of experiments, modeling assumptions, and analysis of results. For this reason, these aspects will be discussed in turn with reference to specific papers when necessary.

The work of Hill et al.,[30] on the styrene acrylonitrile system, is typical of previous research on model discrimination. Ampule experiments were performed over the entire range of feed compositions to conversions of less than 5%. Copolymer composition was measured and used to estimate terminal and penultimate reactivity ratios using instantaneous composition equations. Figure 1 is a plot of the data and the fitted curves. Since the terminal and pentultimate models are nested, an F-test was used to show if the added parameters in the penultimate model provided a significantly better fit of the data.

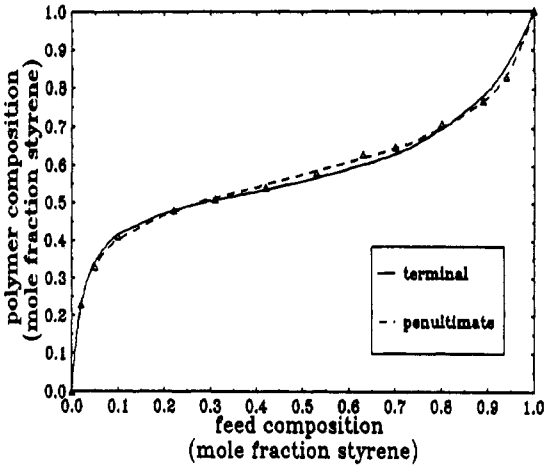

Figure 1. Instantaneous composition data (triangles) from Hill et al.[30] on styrene–acrylonitrile. Terminal model parameter estimates: $\hat{r}_1 = 0.331$ and $\hat{r}_2 = 0.053$. Penultimate parameter estimates: $\hat{r}_{11} = 0.229$, $\hat{r}_{21} = 0.634$, $\hat{r}_{22} = 0.039$, and $\hat{r}_{12} = 0.091$.

These experiments do not seem to be optimal for either parameter estimation or model discrimination. At the beginning of a study very little may be known about the model parameters, but if discrimination is to be achieved, the variance of parameter estimates must be such that prediction variance is of the same order of magnitude or smaller than the average difference between model predictions. Therefore, it is usually necessary to perform some initial parameter estimation experiments. Tidwell and Mortimer[31] used the D-optimal criterion to design terminal model parameter estimation experiments based on the instantaneous composition equation (eq 32). Their results can be summarized in terms of two approximate formulas. If the terminal model parameter estimates are $\hat{r}_1$ and $\hat{r}_2$, the optimal feed compositions are given by

$$f_1 = 2/(2 + \hat{r}_1) \quad (47)$$

and

$$f_1 = \hat{r}_2/(2 + \hat{r}_2) \quad (48)$$

These rules will result in two experimental conditions. For a system with an azeotrope, the feed compositions will be on opposite sides of the azeotrope. Similarly, D-optimal experiments for estimating penultimate reactivity ratios can be designed on the basis of eq 33, as shown in Burke et al.[32] There will be four optimal experiments for estimating the reactivity ratios, two on either side of the azeotropic composition. These feed compositions are very different from the range of compositions from the work of Hill et al.[30] shown in Figure 1, and this is why we state that these experiments do not seem to be optimal for parameter estimation. The experiments also do not seem to be designed for the purpose of model discrimination. As mentioned earlier, the maximum amount of information with respect to discrimination is usually gained by performing experiments where there is a maximum average difference in model predictions. From Figure 1 it can be seen that there is a noticeable difference in model predictions at feed compositions between 0.4 and 0.7 mole fraction of styrene, and some difference at approximately 0.9 mole fraction of styrene. The experiments performed

at other feed compositions provide very little information with respect to discrimination. What the researchers might have been trying to accomplish is parameter estimation and model discrimination in one set of experiments, but since their experiments are not optimal for either purpose, a two stage approach would have made experimentation more efficient, thus reducing the number of experiments required to reach conclusions with a certain level of confidence. It should also be pointed out that a sequential experimental design is an improvement over doing several experiments at once, since information gained from one experiment can be used to plan the next experiment more effectively.

The second problem with experiments like those of Hill et al.[30] involves the equations used to fit the data. For composition data, most researchers use instantaneous equations to fit data and justify this choice by pointing out the low level of conversion in their experiments. For example, with composition data, researchers tend to use the Mayo-Lewis equation[33] or the equivalent penultimate expression. However, even at low conversions there will be some composition drift, and what is being measured is average molar composition, not instantaneous composition. Assuming that feed composition remains constant will introduce some bias onto the results, and given that the differences between terminal and penultimate model predictions are small, this should be corrected by using integrated equations to account for changes in the copolymerization with respect to conversion. It is also important to try to use the actual measured quantity as the independent variable when fitting competing models. For example, in the pulsed laser method, the measured variable is the chain length formed between pulses ν_p. Therefore, the independent variable should be ν_p, not k_p, which is obtained by rearranging eq 46. When k_p is calculated, the original error structure in ν_p is changed. Similarly, it would be best to use ^{13}C NMR peak areas instead of triad fractions to estimate reactivity ratios.

Finally, the data analysis used by some authors could be improved. In some cases, for example the work of Fukuda et al.[34-36] and Ma et al.,[37] the analysis relies heavily on plots of the data. For example, in Fukuda et al.[34] both the terminal and penultimate models were fit to composition data. An examination of the terminal model residuals was then used to show that the terminal model provided an inadequate fit to composition data. Although the residual plot is an invaluable diagnostic tool, it provides qualitative information which may be difficult to interpret when the number of experiments is small relative to the number of parameters. Additional calculations, such as the F-test given in eq 25, are needed to provide quantitative information. It is also interesting that these authors[35-37] modeled composition data using the terminal model and then used the resulting values of $\hat{r}_1$ and $\hat{r}_2$ to fit the terminal and implicit penultimate models to rate data. A better understanding of the system could have been gained by fitting both models to composition and rate data simultaneously. In other cases, the conclusions drawn may have been weakened by the poor experimental design. For example, Hill et al.[30] used an F-test to judge the need for the penultimate model. This is a sound analysis method (in fact, the Buzzi-Ferraris model discrimination method[15] uses F-tests to make decisions about the fit of competing models to the data). Similarly, O'Driscoll and Huang[38] based their conclusions on joint confidence intervals. This again is a sound analysis method because if the extra parameters in the penultimate model are needed to describe the system, certain parameter combinations should not occur in the joint confidence region for the penultimate parameters. However, since the experiments were not optimally designed in either case, the analysis may have failed to recognize a need for the penultimate model in systems where the penultimate effect is less pronounced than that of the styrene–acrylonitrile system studied by Hill et al.[30]

The application of parameter estimation and statistical model discrimination techniques, combined with the use of more rigorous forms of the two models, should improve efforts to model copolymerizations, as will become evident from the remainder of this paper.

Application of Model Discrimination Methods: Our Approach

Steps in the Simulation Program. The review of past work reveals that the systematic application of rigorous statistical model discrimination methods should improve efforts to choose between the terminal and penultimate models for copolymerization. Computer simulations were developed to investigate the benefits of applying model discrimination techniques. The steps used in these computer programs are identical to the steps that would be used to design and analyze real experiments, except that experimental data are generated using a computer simulation.

The model discrimination methods require initial parameter estimates for both the terminal and penultimate models. At the beginning of a study, reliable estimates may not be available. Therefore, the first step in the program is to design parameter estimation experiments. Since the terminal model is nested within the penultimate model, parameter estimation experiments for the penultimate model should lead to good initial estimates for the parameters of both models.

The D-optimal criterion is used to pick four optimal feed compositions $f_{1,1}$, $f_{1,2}$, $f_{1,3}$, and $f_{1,4}$ for estimating the penultimate reactivity ratios. The D-optimal criterion is

$$\varphi = \max[\det(F'F)] \tag{49}$$

where F is the Jacobian at the four feed compositions. Since we are dealing with composition data in this paper, the ith row of the Jacobian contains the derivatives of F_1 with respect to the parameters evaluated at the ith feed composition $f_{1,i}$. For example, the first row of the Jacobian is

$$\left[\frac{dF_1}{dr_{11}}, \frac{dF_1}{dr_{21}}, \frac{dF_1}{dr_{22}}, \frac{dF_1}{dr_{12}}\right] \quad \text{where} \quad f_1 = f_{1,1} \tag{50}$$

Initial guesses of the four reactivity ratios and the four experimental conditions are needed to evaluate the Jacobian. These initial guesses are input to the program, and a simplex optimization technique is used to find the optimal experimental conditions.

Optimizing the D-optimal criterion will produce four optimal experimental conditions, but more than four parameter estimation experiments are simulated. Six parameter estimation experiments are simulated so that we can estimae the four penultimate reactivity ratios and still have two degrees of freedom for the estimate of variance produced by the model s_{pent}^2. Simulating six experiments will also allow us to estimate the two terminal model reactivity ratios and have four degrees of freedom for the terminal model estimate of variance s_{term}^2. The first four parameter estimation experiments are assigned the four feed compositions picked by the D-optimal criterion. The fifth and sixth experiments are assigned the first and third compositions picked by the D-optimal

Table 1. Sources of Kinetic Constants and Monomer and Copolymer Properties

value	STY/MMA	STY/AN	STY/BA
initiator kinetics (AIBN)[a]	Stickler[42]	Stickler[42]	Stickler[42]
homopolymerization rate constants	Kuindersma[43]	Garcia-Rubio[44]	Dubé[45]
monomer reactivity ratios r_{ij}	O'Driscoll and Huang[38]	Hill et al.[30]	Dubé et al.[46]
radical reactivity ratios s_1, s_2	O'Driscoll and Huang[38]	$s_1 = s_2 = 1$	Davis et al.[41]
monomer densities	Kuindersma[43]	Kuindersma[43]	Kuindersma[43]
copolymer density	O'Driscoll and Huang[38]	Note[47]	Dubé et al.[48]
expression for overall k_t	Note[49]	Note[49]	Note[49]

[a] 2,2'-azobis[isobutyronitrile].

criterion, so they are actually replicate experiments. All six experiments are assigned target conversions between 0 and 10 mol % in order to reflect the range of conversions typically used in low conversion composition studies.

Once the conditions for the parameter estimation experiments have been assigned, they are simulated one at a time. On the basis of the initial feed composition for an experiment f_{10} and the target conversion x, the simulation generates an experimental measure of the cumulative composition $\bar{F}_1$. The details of the simulation model are explained in the following subsection.

After the parameter estimation experiments have been simulated, the values of f_{10}, x, and $\bar{F}_1$ are used to estimate the reactivity ratios for both the terminal and penultimate models. The models are fitted using a Levenberg–Marquardt regression, as described by Press et al.[39] or Himmelblau.[40] Regression requires model predictions for both models. For the terminal model, eq 35 is used to obtain a prediction for the final feed composition f_1 based on the values of f_{10} and x. This in turn is used to obtain a prediction of $\bar{F}_1$ from eq 37. Similarly, for the penultimate model, eq 34 is solved numerically to obtain a prediction of the final feed composition f_1 based on the values of f_{10} and x. The prediction f_1 is then used to obtain a prediction of $\bar{F}_1$, from eq 37.

At this point the program picks the first model discrimination experiment. The program can use either the exact entropy,[14] the Hsiang and Reilly,[21] or the Buzzi-Ferraris method[15] to pick a model discrimination experiment. For illustration, the design of experiments using the exact entropy[14] method will be described.

If the exact entropy method is being used, the value of R given by eqs 8, 11, and 12 is maximized to find the optimum values of the feed composition f_{10} and target conversion x. The values of $\hat{y}_i$ and $\hat{y}_j$ in eq 12 are replaced by the terminal and penultimate predictions of cumulative composition $\bar{F}_1$. To find the maximum, R is evaluated at all compositions from $f_{10} = 0.02$ to 0.98 mole fraction of monomer 1 (at intervals of 0.02 mole fraction), and all conversions from 5 to 25 mol % (at intervals of 5 mol %). Note that when R is evaluated, the variance σ^2 is replaced by the penultimate regression estimate s^2_{pent}, which should be smaller than or equal to the estimate based on the terminal model. Note also that if the exact entropy method is being used, model probabilities are needed to evaluate R. The model probabilities are both set to 50% before the six parameter estimation experiments are picked and then are updated using eq 13 on the basis of the data from the six parameter estimation experiments.

The seventh experiment is then simulated at the chosen values of feed composition f_{10} and target conversion x, and the value of cumulative composition $\bar{F}_1$ is recorded. The reactivity ratios are then reestimated using the Levenberg–Marquardt rgression based on the first seven experiments. After the reactivity ratios have been reestimated, the model probabilities are updated. If one of the model probabilities exceeds 95%, then that model is declared best and experimentation stops. If neither of the model probabilities exceeds 95%, then an eighth experiment is picked by maximizing R. Experiments will continue to be designed and analyzed one at a time until one of the model probabilities exceeds 95%, or until the maximum number of experiments (arbitrarily set at 20) is reached.

Simulation of Experimental Data. Our ability to study model discrimination via simulation depends on how well computer generated data duplicate data obtained experimentally. A great deal of effort was expended to develop simulations that could reproduce available experimental data.

The general simulation model was developed as a set of ordinary differential equations. The simulation model was developed as a set of ordinary differential equations. The simulation model has been developed to duplicate low conversion (0–25 mol %) ampule experiments. The equations include a balance on the total moles of radicals, the moles of monomers 1 and 2 being consumed, and the moles of monomer incorporated in the copolymer. The equations also include the change of conversion with time, the change in volume with time, the moments of the molecular weight distribution, and the change in the triad fractions with time.

Three copolymer systems were chosen for the simulation studies. They are styrene–acrylonitrile (STY/AN), styrene–methyl methacrylate (STY/MMA), and styrene–butyl acrylate (STY/BA). One simulation model was developed for each system. The necessary rate constants and monomer and polymer properties have been hard-coded into the model, but reactivity ratios can be input for each simulation run. All equations have been written in terms of penultimate monomer and radical reactivity ratios. But what about STY/MMA which is thought to be best described by an implicit penultimate effect (Fukuda et al.,[35] Davis et al.[41])? This can be accommodated by setting $r_{111} = r_{211} = r_1$ and $r_{222} = r_{122} = r_2$ within the simulation for STY/MMA. In fact, since the terminal model is nested within the penultimate model, the simulation as it is designed can be adapted to reproduce terminal model kinetics, an implicit penultimate effect or an explicit penultimate effect. The sources for kinetic constants and monomer and copolymer properties used in the simulations are listed in Table 1.

Simulations were run to check how well the simulation models predicted experimental data. Figure 2 is a plot of conversion versus time for the system STY/MMA. The symbols are experimental data from Huang[50] and the lines are simulation predictions. Note that no experimental error has been added to the simulation predictions. The simulation seems to adequately predict conversion versus time at all monomer feed compositions. The simulation for STY/BA also adequately predicts conversion versus time data given by Dubé et al.[48] and the simulation for STY/AN adequately predicts the conversion versus time data of Garcia-Rubio.[44] (However, for the purpose of our study, the simulation of STY/AN must be limited to conditions where the reaction is kinetically controlled. The

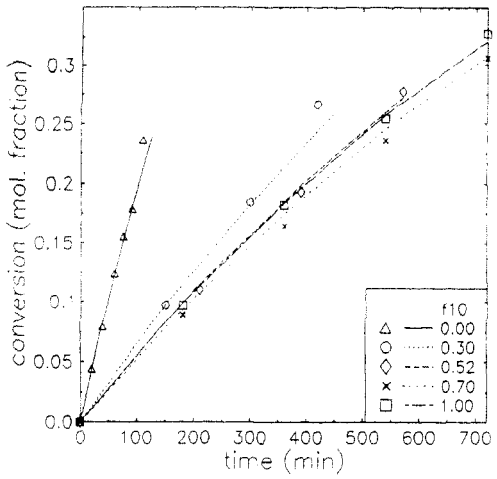

Figure 2. Simulation of conversion versus time data for styrene–methyl methacrylate (Huang[50]). (Since the experiments of Huang[50] were run using the initiator 2,2′-azobis[isobutyric acid methyl ester] (AIBME), kinetic data for AIBME were used to simulate his data. AIBN was used in all other simulations.) Reactivity ratios used: $\hat{r}_{11} = \hat{r}_{21} = 0.472$, $\hat{r}_{22} = \hat{r}_{12} = 0.454$, $\hat{s}_1 = 0.412$, and $\hat{s}_2 = 0.170$. Styrene is monomer one.

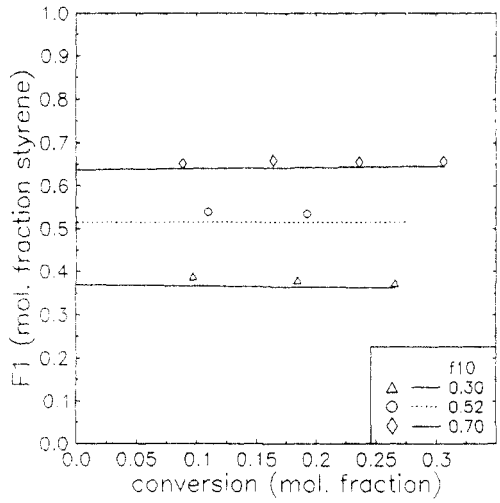

Figure 3. Simulation of cumulative composition versus conversion data (Huang[50]) for styrene–methyl methacrylate. Reactivity ratios are as in Figure 2.

data of Garcia-Rubio[44] at 40 °C show some evidence of diffusion control at low conversions whereas the low conversion data at 60 °C appear to be adequately described by the kinetic models.) Figure 3 is a plot of copolymer composition versus conversion for STY/MMA. The symbols are the experimental data and the lines are the model predictions. Similar results were obtained for STY/BA and STY/AN. Finally, Figure 4 is a plot of triad fractions versus feed composition for STY/AN. The simulation predictions provide a good description of the data of Hill et al.[51]

In order to duplicate experimental results it is also necessary to add experimental error to the predictions of the simulation. To simulate ampule experiments in which cumulative composition is measured by ^{1}H NMR, error should be added to the feed composition, the cumulative composition, and the measured conversion. The literature was again searched to find typical magnitudes of the these errors. Table 2 contains estimates of error in cumulative composition as measured by ^{1}H NMR. A characteristic estimate of error in conversion measurements was reported by Garcia-Rubio et al.[53] at ±2%. An estimate of feed composition error was not found, but from our experience an error of approximately ±1.5 mol % seems reasonable.

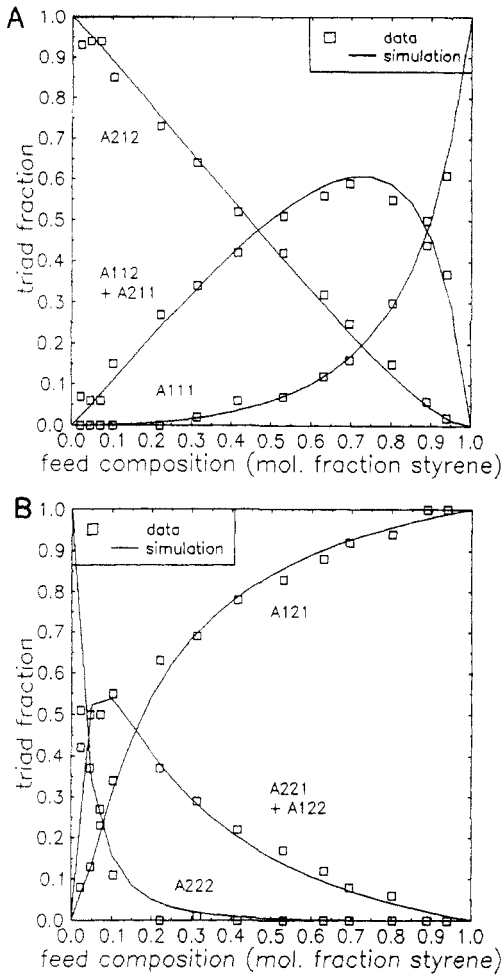

Figure 4. Simulation of triad fraction data (Hill et al.[51]) for styrene–acrylonitrile. Reactivity ratios used: $\hat{r}_{11} = 0.230$, $\hat{r}_{21} = 0.634$, $\hat{r}_{22} = 0.039$, $\hat{r}_{12} = 0.091$, $\hat{s}_1 = 1.00$, and $\hat{s}_2 = 1.00$. Styrene is monomer one and A_{ijk} denotes triad functions.

Table 2. Error in Measuring Cumulative Composition by ^{1}H NMR

error	source
±1.75 wt %	Garcia-Rubio and Ro[52]
±5.00 wt %	Garcia-Rubio et al.[53]
±1.00 wt %	Johnson et al.[54]
±2.00 wt %	Ebdon[55]

Conditions for Simulation Runs. Each run of the program can simulate a total of up to 20 sequentially designed parameter estimation and model discrimination experiments. Using cumulative composition as the measured variable, the simulations were designed to study the application of each model discrimination method (exact entropy, Hsiang-Reilly, and Buzzi-Ferraris) to the three copolymer systems (STY/MMA, STY/AN, and STY/BA). This results in nine combinations of model discrimination method and copolymer system.

For each of these nine combinations, there are several other variables that affect the results of model discrimination experiments. For example, *initial* guesses of *reactivity ratios* are input into the program and used to design the initial parameter estimation experiments. If the initial reactivity ratio estimates are changed, the effect on the initial parameter estimates will affect the overall results of the model discrimination experiments. The size of the *measurement error* added in the simulation will also affect discrimination since increasing the error makes model discrimination more difficult. Finally, the "true" nature of the copolymer system, terminal or penultimate, will affect the results. In real life we can never know the

Table 3
A. Error Levels for STY/AN

level	low	medium	high
feed composition	0.005	0.005	0.005
polymer composition	0.005	0.010	0.015
mole conversion	0.0067	0.0067	0.0067

B. Simulation Parameters for STY/AN

levels	terminal	strong penultimate	small penultimate
r_{11}	0.400	0.230	0.331
r_{21}	0.400	0.634	0.533
r_{22}	0.065	0.039	0.052
r_{12}	0.065	0.091	0.078
s_1	0.700	0.700	1.000
s_2	0.700	0.700	1.000

C. Initial Reactivity Ratio Estimates for STY/AN

levels	poor	neutral	good
$\hat{r}_{11}$	0.600	0.350	0.200
$\hat{r}_{21}$	0.200	0.450	0.600
$\hat{r}_{22}$	0.100	0.060	0.050
$\hat{r}_{12}$	0.050	0.080	0.100
$\hat{r}_1$	0.100	0.300	0.450
$\hat{r}_2$	0.450	0.200	0.100
$\hat{s}_1$	1.000	0.850	0.850
$\hat{s}_2$	1.000	0.850	0.850

"true" model with 100% certainty, but using simulations the level of penultimate effect can be arbitrarily changed by changing the reactivity ratios in the *simulation model*, which is the "true" model for the purpose of the simulation.

These last three variables, the initial reactivity ratios, the level of measurement error, and the simulation model, were investigated by assigning three levels to each variable for each copolymer system. Table 3 shows the levels that were assigned for the study of the system STY/AN.

Table 3A contains the three levels of measurement error. The errors were assumed to be normally distributed and have been expressed in terms of standard deviations. For example, it was estimated from the data in Table 2 that the typical error in cumulative composition is approximately ±0.03 mole fraction and that this represents three standard deviations. Therefore, the medium level of polymer composition error was assigned a standard deviation of 0.03/3 = 0.010 mole fraction. Since this may be too high, a low error level was created with a standard deviation of 0.005 mole fraction and a high error level was created with a standard deviation of 0.015 mole fraction. The error in feed composition was assumed to be approximately ±0.015 mole fraction and was assigned a standard deviation of 0.015/3 = 0.005. The error in conversion was assumed to be approximately ±0.020 mol % and was assigned a standard deviation of 0.0067.

Table 3B contains the reactivity ratios for the three levels of the simulation model. Here the implicit penultimate model will be referred to as the terminal model since the implicit penultimate model cannot be distinguished from the terminal model using composition data alone. The values of the monomer reactivity ratios for the terminal model were set on the basis of the terminal model reactivity ratios given by Garcia-Rubio[44] and Hill et al.[30] The values of the radical reactivity ratios (s_1, s_2) were set arbitrarily to produce what should be a noticeable penultimate effect in future studies of copolymeriztion rate. The second level is a "strong" explicit penultimate effect. The reactivity ratios were set using the penultimate estimates of Hill et al.[30] and the radical reactivity ratios were set to be the same as in the terminal model. This penultimate effect is termed a "strong" penultimate effect because the work of Hill et al.[30] on STY/AN, to the best of our knowledge, is the only study that has shown a penultimate effect on composition data. Thus the differences between reactivity ratios (i.e. $|r_{11} - r_{21}|$) are probably larger than those of other systems. However, this level of penultimate effect is unusually large and it would be useful to investigate cases where the penultimate effect is smaller than that found by Hill et al.[30] for STY/AN. Thus, the third level is a "small" explicit penultimate effect which should be much harder to detect than the strong penultimate effect. For this third level, the reactivity ratios were set so that the differences in reactivity ratios ($r_{11} - r_{21}$ and $r_{22} - r_{12}$) are half of what they are in the "strong" penultimate effect.

Table 3C contains the three levels for initial reactivity ratio estimates. The initial reactivity ratio estimates have been set with reference to the "strong" penultimate case in Table 3B. The first column of Table 3C contains the poor initial estimates. They are called poor estimates because they would be poor reactivity ratio estimates if the "strong" penultimate effect in Table 3B was the "true" simulation model. The third column contains good estimates in that they would be good estimates if the "strong" penultimate effect was the "true" simulation model. The neutral estimates in the second column would not be particularly good nor particularly bad estimates if the "strong" penultimate effect was the "true" model for the system.

The variable levels of STY/MMA and STY/BA were set in a similar manner. For both STY/MMA and STY/BA, the error levels are the same as those given in Table 3A. The simulation parameters for STY/MMA and STY/BA were assigned on the basis of literature values. For STY/MMA the terminal model monomer reactivity ratio were set using the values of O'Driscoll and Huang[38] and the radical reactivity ratios were set on the basis of the same work to produce an implicit penultimate effect in future studies of copolymerization rate. The values for the "strong" and "small" penultimate effects for STY/MMA were then set arbitrarily to produce the same level of penultimate effect as the "strong" and "small" levels for STY/AN in Table 3B. Similarly, for STY/BA the terminal model monomer reactivity ratios were set using the values of Dubé et al.[46] and the radical reactivity ratios were set using the dilatometric data of Dubé et al.[48] to produce an implicit penultimate effect in future studies of copolymerization rate. The values for the "strong" penultimate effect were set using the penultimate reactivity ratios of Dubé et al.[46] and the values for the "small" penultimate effect were set arbitrarily to produce a penultimate effect which is about half the magnitude of the "strong" penultimate effect.

The most efficient way to study the effect of the three variables is by designing a set of simulation runs for each combination of model discrimination method and copolymer system in the form of a 3^3 factorial experiment. Table 4 shows the combinations of simulation model, initial reactivity ratio estimates, and error level for each run. For example, the seventh run would be set up with the terminal model as the "true" simulation model, "good" initial reactivity ratio estimates, and the low level of measurement error.

In order to compare the results of different simulation runs, a history file is created for each simulation. The history file contains a record of the feed composition chosen for each experiment, the value of the simulated composition measurement $\bar{F}_1$, and the estimated reactivity ratios for both the terminal and penultimate models. The file will also contain information on which model was chosen

Table 4. Experimental Design for Simulation Runs[a]

run	simulation model	initial reactivity ratio estimates	error level
1	terminal	poor	low
2	terminal	poor	medium
3	terminal	poor	high
4	terminal	neutral	low
5	terminal	neutral	medium
6	terminal	neutral	high
7	terminal	good	low
8	terminal	good	medium
9	terminal	good	high
10	strong penultimate	poor	low
11	strong penultimate	poor	medium
12	strong penultimate	poor	high
13	strong penultimate	neutral	low
14	strong penultimate	neutral	medium
15	strong penultimate	neutral	high
16	strong penultimate	good	low
17	strong penultimate	good	medium
18	strong penultimate	good	high
19	small penultimate	poor	low
20	small penultimate	poor	medium
21	small penultimate	poor	high
22	small penultimate	neutral	low
23	small penultimate	neutral	medium
24	small penultimate	neutral	high
25	small penultimate	good	low
26	small penultimate	good	medium
27	small penultimate	good	high

[a] The values used for the simulation model, the initial reactivity ratios, and the error level are given for STY/AN in Table 3.

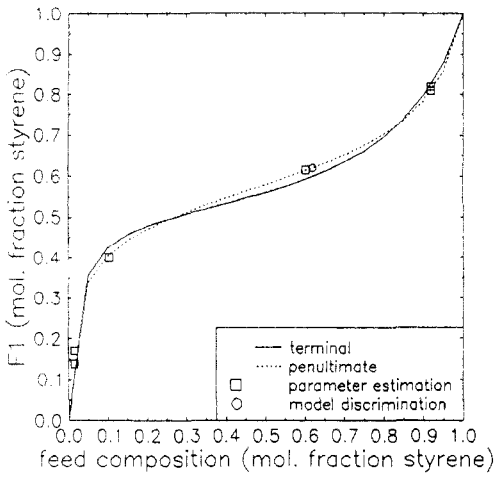

Figure 5. Example plot of the simulation results for the application of the Buzzi-Ferraris method to the system styrene–acrylonitrile. The squares are parameter estimation experiments designed using the D-optimal criterion. The circle is the model discrimination experiment designed using the Buzzi-Ferraris method.

as the "best" model for the system. If the Buzzi-Ferraris method is being used, the file contains the upper tail probability of the F-statistic for each experiment. If the exact entropy or the Hsiang and Reilly method is being used, the file contains the updated model probabilities for each experiment.

Results and Discussion

The results of the simulations provide a clear comparison of experiments designed using model discrimination methods and those that have previously appeared in the literature. Figure 5 shows the results of run 13 (conditions for this run are given in Table 4) in which the Buzzi-Ferraris method was applied to the system STY/AN. In this run, the simulation model was the strong penultimate effect with model parameters $r_{11} = 0.230$, $r_{21} = 0.634$, $r_{22} = 0.039$, and $r_{12} = 0.091$. The initial parameter estimates were $\hat{r}_{11} = 0.350$, $\hat{r}_{21} = 0.450$, $\hat{r}_{22} = 0.060$, and $\hat{r}_{12} = 0.080$ and the standard deviation of the error in cumulative composition was 0.005 mole fraction. The conditions for this simulation run are almost identical to the conditions under which the experiments in Figure 1 were performed. In this simulation run the penultimate model was chosen as the best model for composition data at 98.3% confidence using just 7 experiments, whereas in the work of Hill et al.[30] shown in Figure 1, 14 experiments were performed and the penultimate model was chosen as best with 99.5% confidence. In general, by using the Buzzi-Ferraris method on the system STY/AN, it was possible to discriminate between the terminal and penultimate models in fewer experiments than would be required if experiments were evenly spread over the full range of feed compositions. Thus it seems it is possible to reduce the number of experiments required to identify the best model for the system by using a combination of parameter estimation and model discrimination experiments.

Table 5

simulation model	model chosen as best at 95% confidence (number of simulation runs) terminal	penultimate	neither
A. Application of the Buzzi-Ferraris Method to STY/MMA			
terminal	4	1	4
strong penultimate	1	7	1
small penultimate	1	5	3
B. Application of the Buzzi-Ferraris Method to STY/AN			
terminal	2	2	5
strong penultimate	0	9	0
small penultimate	1	6	2
C. Application of the Buzzi-Ferraris Method to STY/BA			
terminal	1	1	7
strong penultimate	2	6	1
small penultimate	1	5	3

Comparison of Model Discrimination Methods. Table 5 gives a summary of the simulations in which the Buzzi-Ferraris method was used to discriminate between the terminal and penultimate models. For each copolymer system, a total of 27 simulation runs were performed. The conditions for these simulations are shown in Table 4. In runs 1–9 the "true" model was the terminal model, in runs 10–18 the "true" simulation model was the strong penultimate effect, and in runs 19–27 the simulation model was the small penultimate effect. Table 5A shows the results for the system STY/MMA. For the first nine runs, in which the "true" simulation model was the terminal model, the terminal model was picked as best in four of the nine runs, the penultimate model was picked in one of the nine runs, and the model discrimination method picked neither model as best in four of the nine runs (i.e. the model discrimination method could not discriminate at 95% in a total of 20 experiments). Similarly, for the second nine runs in which the simulation model was the strong penultimate model, the terminal model was picked as best in one run, the penultimate model was picked as best in seven runs, and neither model was picked in one run. Even in the last nine runs, in which the simulation model was the small penultimate effect, the penultimate model was picked in five of the nine runs. These trends are similar to those given in Table 5B for STY/AN and in Table 5C for STY/BA.

These results are very encouraging. When the "true" simulation model was the strong penultimate model, the penultimate model was picked as the best model in 80% of the runs, and when the simulation model was the small

penultimate model, the penultimate model was picked as best model in 63% of the runs. This indicates that properly designed experiments can be used to find penultimate effects in composition data if they exist, even when the penultimate effects are smaller than the strong penultimate effect found by Hill et al.[30] for STY/AN.

The discouraging aspect of these results is that there are several runs in which the simulation model was not chosen as the best model for the system.

There were three runs in which the "true" model was the strong penultimate model and yet the terminal model was picked as the best model for the system. Two of these runs were performed with the "high" level of measurement error. The "high" level of measurement error in cumulative composition is actually higher than all but one of the values reported in Table 2. Therefore, it is not surprising that the "high" level of measurement error causes some problems in model discrimination. The third run was a run for the system STY/BA in which the simulation parameters were $r_{11} = 0.551$, $r_{21} = 0.937$, $r_{22} = 0.225$, and $r_{12} = 0.130$. In several of the simulations for the system STY/BA, there were problems obtaining good parameter estimates. These problems involved values of $\hat{r}_{21}$ that were greater than 1. It seems that when one of the reactivity ratios, such as r_{21}, has a true value close to 1.0, it is more difficult to accurately estimate the reactivity ratios and thus more difficult to correctly discriminate between the terminal and penultimate models.

There were also four runs in which the simulation model was the terminal model and the penultimate model was falsely picked as the best model for the system. Examination of the results showed that in three of the four runs, $s_{pent}^2 < \sigma^2$, which means the penultimate estimate of variance is underestimating the actual level of error in the simulated measurements. In these cases, the F-statistic given in eq 25 is biased toward the penultimate model because $SSR_{pent}/(n - k_{pent})$ underestimates the true experimental error. Theoretically, if the terminal model is the true model of the system, the lower limit on both s_{term}^2 and s_{pent}^2 is the error variance σ^2. However, when the terminal model is the true model and the number of experiments is small relative to the number of parameters in both models, the penultimate model which has two extra reactivity ratios may begin to model some of the random error. Thus, it is possible for s_{pent}^2 to be less than σ^2. The Buzzi-Ferraris method would be more reliable if we could overcome this difficulty. One solution to this problem is to change the form of the criterion in eq 25 to account for the probability that s_{pent}^2 will be less than σ^2. Other possibilities include increasing the confidence level required to accept the penultimate model as best and performing replicate experiments at each feed composition in order to obtain an independent estimate of experimental error. For example, when simulations were repeated using two independent replicates of each model discrimination experiment, only one run in 27 remained in which the penultimate model was falsely picked as the best model. One run in 27 represents a 4% failure, which is roughly what would be expected when using a 95% confidence level.

Table 6 gives a summary of the results for the simulations in which the exact entropy method was used to design and analyze model discrimination experiments. Table 6A shows the results for the system STY/MMA. Overall, the results for the exact entropy criterion were not as promising as those with the Buzzi-Ferraris criterion. For the system STY/AN (Table 6B), the exact entropy method was able to correctly identify the penultimate model as the best model in only six of the eighteen simulation runs. The exact entropy criterion had the tendency to pick the terminal model as the best model for the system even if the strong penultimate model was used to generate experimental data. Why is the exact entropy method less likely to identify the correct model than the Buzzi-Ferraris method? Figure 6 shows the results of run 13 for the system STY/AN when the exact entropy method was used to design model discrimination experiments. The results obtained with the Buzzi-Ferraris criterion under the same conditions are given in Figure 5. The figures appear to be identical, and this is not a printing error. In both Figures 5 and 6, the first six experiments are parameter estimation experiments and the model discrimination experiment is performed at the same feed composition. Therefore, the main difference between the Buzzi-Ferraris and the exact entropy method seems to be in the analysis of the experimental data. The use of a Bayesian analysis in the exact entropy method must be less able to detect small differences in the fit of the terminal and penultimate models to composition data than the F-statistic used in the Buzzi-Ferraris method.

In six of the runs in which the terminal model was the "true" model, the exact entropy method falsely picked the penultimate model as the best model for the system. This seems to result from the fact that in these runs s_{pent}^2 is less than σ^2. The penultimate estimate s_{pent}^2 was used in the evaluation of model likelihoods given by eq 14. It was used because it should be the best estimate of error variance when the penultimate model is the true model, and it should be equal to the terminal model estimate when the

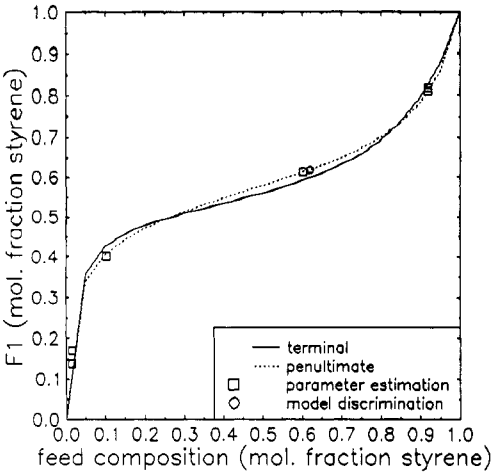

Figure 6. Example plot of the simulation results for the application of the exact entropy method to the system styrene–acrylonitrile.

Table 6

	model chosen as best at 95% confidence (number of simulation runs)		
simulation model	terminal	penultimate	neither
A. Application of the Exact Entropy Method to STY/MMA			
terminal	7	2	0
strong penultimate	9	0	0
small penultimate	8	1	0
B. Application of the Exact Entropy Method to STY/AN			
terminal	5	2	2
strong penultimate	3	4	2
small penultimate	5	2	2
C. Application of the Exact Entropy Method to STY/BA			
terminal	7	2	0
strong penultimate	9	0	0
small penultimate	9	0	0

Table 7

simulation model	model chosen as best at 95% confidence (number of simulation runs) terminal	penultimate	neither
A. Application of the Hsiang and Reilly Method to STY/MMA			
terminal	9	0	0
strong penultimate	7	1	1
small penultimate	9	0	0
B. Application of the Hsiang and Reilly Method to STY/AN			
terminal	5	3	1
strong penultimate	4	5	0
small penultimate	7	2	0
C. Application of the Hsiang and Reilly Method to STY/BA			
terminal	9	0	0
strong penultimate	6	2	1
small penultimate	9	0	0

terminal model is the true model of the system. When s_{pent}^2 is less than σ^2, both the terminal and penultimate likelihoods are overestimated by some factor that depends on the size of the residual for the last experiment. The model with the smaller residual will experience the largest overestimation of the model likelihood. Thus, when s_{pent}^2 is less than σ^2, the exact entropy method will tend to pick one of the models prematurely. This will include simulations in which the penultimate model is picked as best even though the terminal model was used to generate experimental data. In order to improve the performance of the exact entropy criterion, it seems necessary to obtain a reliable estimate of experimental error from previous experimental studies or by performing replicate experiments at each feed composition chosen.

Finally, Table 7 summarizes the results for the simulations in which the Hsiang and Reilly method was used to discriminate between the terminal and penultimate models. The results are very similar to those obtained using the exact entropy method in Table 6. Therefore, the difference between the Hsiang and Reilly and Buzzi-Ferraris methods could be similar to the difference between the exact entropy and Buzzi-Ferraris methods. The difference could be due to the fact that the F test used in the Buzzi-Ferraris method is more sensitive to small differences in the models than the Bayesian analysis. However, when the results of the Hsiang and Reilly method are compared to the Buzzi-Ferraris method, differences are also found in the feed compositions chosen for model discrimination experiments. At this point, it is not certain whether the poor performance of the Hsiang and Reilly method compared to the Buzzi-Ferraris method is due to the experimental design criterion or the data analysis. The one noticeable difference between the exact entropy and the Hsiang and Reilly results is that the Hsiang and Reilly method uses parameter probability distributions to account for the uncertainty in the parameter estimates. Since the Hsiang and Reilly method updates parameter probability distributions, it requires more experimental data to discriminate between models. The Hsiang and Reilly method typically required 14–16 experiments to pick either the terminal or the penultimate model as the best model for the system, while the exact entropy method was able to make a decision after 6–8 experiments. Therefore, for composition data, it does not seem that the use of parameter probability distributions provided any additional benefits.

It is interesting to note that for the system STY/AN (Table 7B) there were three runs in which the Hsiang and Reilly method falsely picked the penultimate model as best when the terminal model was used to generate experimental data. This is similar to problems observed with both the Buzzi-Ferraris and exact entropy methods. In the cases of the Buzzi-Ferraris and exact entropy methods, the penultimate model was falsely picked because the penultimate estimate of variance was underestimating the true error variance. However, in the case of the Hsiang and Reilly method, it is not certain whether the problem was caused by underestimation of the true error variance or whether the number of discrete levels used to create the probability distributions was too small to accurately represent the distributions.

Comparison of Copolymer Systems. There are also differences in the way model discrimination methods worked with the systems STY/MMA, STY/AN, and STY/BA. For example, when the strong penultimate effect was used to generate data for all three copolymer systems, the three model discrimination methods worked best with the system STY/AN. Is this because the model discrimination techniques work differently for different copolymer systems, or because the reactivity ratios used to generate the experimental data are not simulating the same level of penultimate effect for the three copolymer systems? It seems to be a bit of both.

For the system STY/BA, one of the reactivity ratios is close to 1 for both the terminal and penultimate models. This seemed to make it more difficult to accurately estimate reactivity ratios and thus it affected the ability of the model discrimination methods to discriminate between the terminal and penultimate models. In general, model discrimination methods are expected to be less effective when applied to copolymer systems which pose special difficulties in reactivity ratio estimation, but this must be recognized as a parameter estimation problem, not a problem with the model discrimination methods.

For the system STY/MMA, there were no unusual problems in estimating the reactivity ratios. The differences between the simulations for STY/MMA and STY/AN seemed to be caused by the inability to generate the same magnitude of penultimate effect for both systems.

Concluding Remarks

This paper summarizes the application (at the simulation level) of model discrimination methods to three styrene systems using copolymer composition as the measured variable.

The results show that the systematic application of model discrimination experiments can reduce the number of experiments required to choose between the terminal and penultimate models for copolymerization compared to designs in which experiments are spread over the entire range of feed compositions. The other notable result is that the Buzzi-Ferraris method is capable of detecting much smaller penultimate effects (when they exist) than those found by Hill et al.[30] for the system STY/AN. Thus, it may be necessary to reconsider the usefulness of composition data for discriminating between the terminal and penultimate models.

There are differences in the performance of the three model discrimination methods tested. The Buzzi-Ferraris method was able to identify cases in which the penultimate model was the best model for the system. When the strong penultimate model was used to generate data, the Buzzi-Ferraris method chose the penultimate model as best in 80% of the runs. Yet, the exact entropy and Hsiang and Reilly methods generally picked the terminal model as best, even when the strong penultimate model was used to simulate data. The difference between the Buzzi-Ferraris and exact entropy method seems to be that the

Bayesian analysis used in the exact entropy method is not as sensitive to small differences in the models as the F-statistic used in the Buzzi-Ferraris method. The difference between the Buzzi-Ferraris and Hsiang and Reilly methods includes differences in the data analysis as well as in the design of experiments.

One similarity between the three methods is how they respond to poor estimates of measurement error. When the terminal model is used to simulate experimental data, the penultimate estimate of error may underestimate the true error. If the penultimate estimate of error is then used as the estimate of measurement error in the data analysis, the penultimate model may be falsely picked as the best model for the system. This problem occurred with the Buzzi-Ferraris and exact entropy methods, and may have been a factor in the application of the Hsiang and Reilly method. Performing replicate experiments improves the penultimate estimate of variance and may help to eliminate this problem. If performing replicate experiments is not possible, other suggestions have been made in our paper that will address this problem.

The first set of results has clearly shown that the application of statistical model discrimination methods should lead to improvements in modeling copolymer systems. The results have also highlighted the strengths and weaknesses of existing model discrimination methods. Future work will show if the same trends hold true when triad or rate data are used to discriminate between the terminal and penultimate models.

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$$k_t = F_1 k_{t1} + F_2 k_{t2}$$

In the cases of STY/MMA and STY/BA, the overall termination rate constant was calculated using a different composition-weighted average. The overall termination rate was calculated using

$$k_t = \frac{k_{t1} r_{21} f_1^2 \dfrac{(r_{11} f_1 + f_2)}{(r_{21} f_1 + f_2)} + 2\sqrt{k_{t1} k_{t2}} f_1 f_2 + k_{t2} r_{12} f_2^2 \dfrac{(r_{22} f_2 + f_1)}{(r_{12} f_2 + f_1)}}{r_{21} f_1^2 \dfrac{(r_{11} f_1 + f_2)}{(r_{21} f_1 + f_2)} + 2 f_1 f_2 + r_{12} f_2^2 \dfrac{(r_{22} f_2 + f_1)}{(r_{12} f_2 + f_1)}}$$

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